

A BASIC CLASS OF SYMMETRIC ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

MOHAMMAD MASJED-JAMEI AND IVÁN AREA

ABSTRACT. By using a generalization of Sturm-Liouville problems in discrete spaces, a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters, which generalizes all classical discrete symmetric orthogonal polynomials, is introduced. The standard properties of these polynomials, such as a second order difference equation, an explicit form for the polynomials, a three term recurrence relation and an orthogonality relation are presented. It is shown that two hypergeometric orthogonal sequences with 20 different weight functions can be extracted from this class. Moreover, moments corresponding to these weight functions can be explicitly computed. Finally, a particular example containing all classical discrete symmetric orthogonal polynomials is studied in detail.

1. INTRODUCTION

Some special functions of mathematical physics such as classical orthogonal polynomials and cylindrical functions [29], are solutions of a differential equation of hypergeometric type [28, 29, 32]

$$(1) \quad \sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0,$$

and extendible by changing equation (1) to a difference equation of the form

$$(2) \quad \tilde{\sigma}(x(s)) \frac{\Delta}{\nabla x_1(s)} \left[\frac{\nabla y(s)}{\nabla x(s)} \right] + \frac{\tilde{\tau}(x(s))}{2} \left[\frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda y(s) = 0,$$

where

$$\Delta x(s) = x(s+1) - x(s), \quad \nabla x(s) = x(s) - x(s-1), \quad \frac{\Delta}{\Delta x(s)} f(s) = \frac{f(s+1) - f(s)}{x(s+1) - x(s)},$$

$\tilde{\sigma}(x(s))$ and $\tilde{\tau}(x(s))$ are polynomials of degree at most two and one, respectively, in $x(s)$, λ is a constant, and $x_1(s) = x(s+1/2)$.

Date: October 11, 2012.

2010 *Mathematics Subject Classification.* Primary: 42C05, 33E30, 33C47 Secondary: 33C45, 33C20.

Key words and phrases. Extended Sturm-Liouville theorem for symmetric functions of a discrete variable, Classical symmetric orthogonal polynomials of a discrete variable, Hypergeometric series, Symmetric Kravchuk and Hahn-Eberlein polynomials.

Acknowledgments. The work of M. Masjed-Jamei has been supported by a grant from “Iran National Science Foundation” No. 91002576 and the work of I. Area has been partially supported by the Ministerio de Ciencia e Innovación of Spain under grants MTM2009-14668-C02-01 and MTM2012-38794-C02-01, co-financed by the European Community fund FEDER. The referee and handling editor deserve special thanks for careful reading and many useful comments and suggestions which have improved the manuscript. Dedicated to Prof. A. Ronveaux on the occasion of his 80th Birthday.

The difference equation (2), which is obtained by approximating the differential equation (1) on a non-uniform lattice, is of much importance [32] as its particular solutions have been applied in quantum mechanics, theory of group representations and especially computational mathematics, where one can point to the Clebsch-Gordan and Racah coefficients with wide applications in atomic and nuclear spectroscopy. There exist different approaches for the analysis of orthogonal polynomials of a discrete variable running from the classical references [8, 17] to the recent monograph [16], which is a basic reference on orthogonal polynomials.

Also there exists a number of numerical and symbolic methods for solving hypergeometric equations of type (1) or (2), which are of interest in applications, particularly for cases containing symmetric solutions, such as resolution of the Gibbs phenomenon [10, 11], Fourier-Kravchuk transform used in Optics [6], approximation of harmonic oscillator wave functions [4], tissue segmentation of human brain MRI through preprocessing [1], reconstructions for electromagnetic waves in the presence of a metal nanoparticle [27], efficient determination of the critical parameters and the statistical quantities for Klein-Gordon and sine-Gordon equations with a singular potential [7], image representation [33, 14] and quantitative theory for the lateral momentum distribution after strong-field ionization [9].

The main aim of this paper is to introduce a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters, which is the polynomial solution of a symmetric generalization of equation (2) on the uniform lattice $x(s) = s$. Computational aspects of these new polynomials are described in detail giving their explicit representation as well as the three-term recurrence relation they satisfy. A full classification of weight functions and orthogonality supports is given together with computing the moments of the aforesaid weights. From this class all classical symmetric orthogonal polynomials of a discrete variable can be recovered (section 6), and its limit relation with the continuous type of generalized classical symmetric orthogonal polynomials is given (see remark 2).

A regular Sturm-Liouville problem of continuous type is a boundary value problem in the form

$$(3) \quad \frac{d}{dx} \left(k(x) \frac{dy_n(x)}{dx} \right) + (\lambda_n \varrho(x) - q(x)) y_n(x) = 0 \quad (k(x) > 0, \varrho(x) > 0),$$

which is defined on an open interval (a, b) , and has the boundary conditions

$$(4) \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0,$$

where α_1, α_2 and β_1, β_2 , are given constants and $k(x)$, $k'(x)$, $q(x)$, and $\varrho(x)$ in (3) are to be assumed continuous for $x \in [a, b]$. If one of the boundary points a and b is singular (i.e. $k(a) = 0$ or $k(b) = 0$), the problem is transformed to a singular Sturm-Liouville problem.

Let y_n and y_m two eigenfunctions of the operator $D(k(x)D) - q(x)I$, where D is the standard derivative operator. According to Sturm-Liouville theory [29], they are orthogonal with respect to the weight function $\varrho(x)$ under the given conditions (4) and satisfy the orthogonality relation

$$\int_a^b \varrho(x) y_n(x) y_m(x) dx = \left(\int_a^b \varrho(x) y_n^2(x) dx \right) \delta_{n,m}.$$

Many of special functions are orthogonal solutions of a regular or singular Sturm-Liouville problem having the symmetry property ($\phi_n(-x) = (-1)^n \phi_n(x)$) so that

have found valuable applications in physics and engineering, as already mentioned. In [20], the classical equation (3) is symmetrically extended as follows.

Theorem 1. [20] *Let $\phi_n(-x) = (-1)^n \phi_n(x)$ be a sequence of symmetric functions satisfying the equation*

$$(5) \quad A(x)\phi_n''(x) + B(x)\phi_n'(x) + (\lambda_n C(x) + D(x) + \sigma_n E(x))\phi_n(x) = 0,$$

where

$$(6) \quad \sigma_n = \frac{1 - (-1)^n}{2} = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$

and λ_n is a sequence of constants. If $A(x)$, $(C(x) > 0)$, $D(x)$ and $E(x)$ are even functions and $B(x)$ is odd then

$$\int_{-\nu}^{\nu} \varrho^*(x)\phi_n(x)\phi_m(x)dx = \left(\int_{-\nu}^{\nu} \varrho^*(x)\phi_n^2(x)dx \right) \delta_{n,m},$$

where

$$(7) \quad \varrho^*(x) = C(x) \exp \left(\int \frac{B(x) - A'(x)}{A(x)} dx \right) = \frac{C(x)}{A(x)} \exp \left(\int \frac{B(x)}{A(x)} dx \right).$$

The weight function defined in (7) must be positive and even on $[-\nu, \nu]$ and the function

$$A(x)K(x) = A(x) \exp \left(\int \frac{B(x) - A'(x)}{A(x)} dx \right) = \exp \left(\int \frac{B(x)}{A(x)} dx \right)$$

must vanish at $x = \nu$, i.e. $A(\nu)K(\nu) = 0$. In this way, since $K(x) = \varrho^*(x)/C(x)$ is an even function so $A(-\nu)K(-\nu) = 0$ automatically.

Using this theorem, many symmetric special functions of continuous type have been generalized in [19, 20, 21, 22, 24, 25, 26].

Orthogonal functions of a discrete variable can similarly be solutions of a regular or singular Sturm-Liouville problem of discrete type in the form [15]

$$(8) \quad \Delta(K^*(x)\nabla y_n(x)) + (\lambda_n w(x) - q^*(x))y_n(x) = 0 \quad (K^*(x) > 0, w(x) > 0),$$

where

$$\Delta f(x) = \nabla f(x+1) = f(x+1) - f(x),$$

and (8) satisfies a set of discrete boundary conditions like (4). This means that if $y_n(x)$ and $y_m(x)$ are two eigenfunctions of difference equation (8), they are orthogonal with respect to the weight function $\varrho^*(x)$ on a discrete set [28].

Recently in [23] we have presented the following theorem by which one can generalize usual Sturm-Liouville problems with symmetric solutions in discrete spaces. As a very important consequence of this theorem, we can introduce a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters.

Theorem 2. [23] *Let $\phi_n(-x) = (-1)^n \phi_n(x)$ be a sequence of symmetric functions that satisfying the difference equation*

$$(9) \quad A(x)\Delta\nabla\phi_n(x) + (A(-x) - A(x))\Delta\phi_n(x) + (\lambda_n C(x) + D(x) + \sigma_n E(x))\phi_n(x) = 0,$$

where, as usual, $\Delta\nabla = \Delta - \nabla$. If $A(x)$ is a free real function and $(C(x) > 0)$, $D(x)$ and $E(x)$ are even functions then

$$\sum_{x=\alpha}^{\beta-1} W^*(x) \phi_n(x) \phi_m(x) = \left[\sum_{x=\alpha}^{\beta-1} W^*(x) \phi_n^2(x) \right] \delta_{n,m}$$

where

$$(10) \quad W^*(x) = C(x)W(x)$$

and $W(x)$ is the solution of the Pearson difference equation

$$\Delta [A(x)W(x)] = [A(-x) - A(x)] W(x),$$

which is equivalent to

$$(11) \quad \frac{W(x+1)}{W(x)} = \frac{A(-x)}{A(x+1)}.$$

Moreover, the weight function defined in (10) must be even over one of the four following symmetric counter sets

- i) $S_1 = \{-a-n, -a-n+1, \dots, -a-1, -a, a, a+1, \dots, a+n\}$, $a \in \mathbf{R}$,
- ii) $S_2 = S_1 \cup \{0\}$ (as any odd function is equal to zero at $x=0$),
- iii) $S_3 = \{\dots, -a-n, -a-n+1, \dots, -a-1, -a, a, a+1, \dots, a+n, \dots\}$ (an infinite set)
- iv) $S_4 = S_3 \cup \{0\}$,

and the function $A(x)W(x)$ must also vanish at $x = \alpha$ and $x = 1 - \beta$, where $[\alpha, \beta - 1] \in \{S_1, S_2, S_3, S_4\}$.

2. A BASIC CLASS OF SYMMETRIC ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE USING THEOREM 2

As a special case of equation (5), the following differential equation is defined in [19]:

$$(12) \quad x^2(p^*x^2 + q^*)\Phi_n''(x) + x(r^*x^2 + s^*)\Phi_n'(x) - (n(r^* + (n-1)p^*)x^2 + \sigma_n s^*)\Phi_n(x) = 0.$$

Here one of the basic solutions is the symmetric class of orthogonal polynomials denoted by

$$S_n^* \left(\begin{array}{cc|c} r^* & s^* & t \\ p^* & q^* & \end{array} \right) = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \times \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{(2i + (-1)^{n+1} + 2[n/2])p^* + r^*}{(2i + (-1)^{n+1} + 2)q^* + s^*} \right) x^{n-2k}.$$

By referring to theorem 1, we observe in (12) that $A(x) = x^2(p^*x^2 + q^*)$ is a polynomial of degree at most four, $C(x) = x^2$ is a symmetric quadratic polynomial, $D(x) = 0$, and $E(x) = s^*$. Since discrete orthogonal polynomials have a direct relationship with continuous polynomials, motivated by (12), we suppose in the main equation (9) that

$$(13) \quad A(x) = \sum_{i=0}^4 a_i x^i, \quad C(x) = c_2 x^2 + c_0, \quad D(x) = 0, \quad E(x) = e_0,$$

though there may exist some other possible cases.

By noting the assumptions (13), we are now interested in obtaining a symmetric orthogonal polynomial solution. Hence, let

$$(14) \quad \phi_n(x) = x^n + \delta_n x^{n-2} + \cdots,$$

satisfy a three term recurrence relation as

$$(15) \quad \phi_{n+1}(x) = x\phi_n(x) - \gamma_n\phi_{n-1}(x), \quad (\text{with } \phi_0(x) = 1, \phi_1(x) = x).$$

From (9), (13) and (14), equating the coefficient in x^{n+2} gives

$$(16) \quad \lambda_n = \frac{n(2a_3 - a_4(n-1))}{c_2},$$

provided that $c_2 \neq 0$ and also $|a_3| + |a_4| \neq 0$.

By using the eigenvalue λ_n in (16) and equating the coefficient in x^n we obtain

$$\begin{aligned} \delta_n = \{ & 6c_2(4a_1n - 2a_2(n-1)n + e_0((-1)^n - 1)) \\ & + a_4(n-1)n(12c_0 - c_2(n-3)(n-2)) \\ & + 4a_3n(c_2(n-2)(n-1) - 6c_0) \} / \{ 24c_2(a_4(3-2n) + 2a_3) \}. \end{aligned}$$

Also from (14) and (15) we have

$$x^{n+1} + \delta_{n+1}x^{n-1} + \cdots = x(x^n + \delta_n x^{n-2} + \cdots) - \gamma_n(x^{n-1} + \delta_{n-1}x^{n-3} + \cdots),$$

which implies

$$(17) \quad \gamma_n = \delta_n - \delta_{n+1}.$$

Therefore, in order that $\phi_n(x)$ is a solution of (9), the following extra conditions must be considered for the initial values of n :

$$e_0 = 2a_1 - \frac{2a_3c_0}{c_2}, \quad a_0 = \frac{c_0((2a_3 - a_4)c_0 + (a_2 - 2a_1)c_2)}{c_2^2}, \quad c_2 = -4c_0.$$

As a conclusion, we get

$$(18) \quad \begin{cases} A(x) = a_4x^4 + a_3x^3 + a_2\left(x^2 - \frac{1}{4}\right) + a_1\left(x + \frac{1}{2}\right) + \frac{a_3}{8} - \frac{a_4}{16}, \\ B(x) = A(-x) - A(x) = -2x(a_3x^2 + a_1), \\ C(x) = c_0(1 - 4x^2), \\ D(x) = 0, \\ E(x) = \frac{1}{2}(4a_1 + a_3), \\ \lambda_n = \frac{n(a_4(n-1) - 2a_3)}{4c_0}, \end{cases}$$

and

$$(19) \quad \begin{aligned} \delta_n = \{ & 12a_1(2n + (-1)^n - 1) + 3a_3((-1)^n - 1) \\ & + n(-12a_2(n-1) + 2a_3(2(n-3)n + 7) \\ & - a_4(n-1)((n-5)n + 9)) \} / \{ 24a_4(3-2n) + 48a_3 \}. \end{aligned}$$

For simplicity, if we set

$$a_4 = 2a, \quad a_3 = a + 2b, \quad a_2 = b + 2c, \quad a_1 = c + 2d,$$

then (18) changes to

$$\begin{cases} A(x) = (2x+1)(ax^3+bx^2+cx+d), \\ B(x) = -2x((a+2b)x^2+c+2d), \\ C(x) = \frac{1}{4}-x^2, \\ D(x) = 0, \\ E(x) = \frac{1}{2}(a+2b+4c+8d), \\ \lambda_n = 2n(an-2(a+b)) \text{ for } |a|+|b| \neq 0. \end{cases}$$

Hence the following difference equation appears

$$(20) \quad (2x+1)(ax^3+bx^2+cx+d)\Delta\nabla\phi_n(x) - 2x(x^2(a+2b)+c+2d)\Delta\phi_n(x) \\ + \left(2n(an-2(a+b))\left(\frac{1}{4}-x^2\right) + \sigma_n\left(\frac{a}{2}+b+2c+4d\right)\right)\phi_n(x) = 0.$$

Since the polynomial solution of equation (20) is symmetric, we use the notation

$$\phi_n(x) = S_n \left(\begin{array}{cc|c} a & b & x \\ c & d & \end{array} \right),$$

for mathematical formulae and $S_n(a, b, c, d; x)$ in the sequel. This means that from now we deal with just one characteristic vector $\vec{V} = (a, b, c, d)$ for any given sub-case.

If the polynomial sequence $\{S_n(a, b, c, d; x)\}_{n \geq 0}$ satisfies a three term recurrence relation of type (15) then by referring to (17) and (19) we obtain

Proposition 1. *The coefficient γ_n in the three term recurrence relation (15) is a rational expression in n where the numerator is a polynomial of degree 4 and the denominator is a polynomial of degree 2, given by*

$$(21) \quad \gamma_n = \gamma_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\sum_{i=0}^4 K_i(a, b, c, d)n^i}{32(b-a(n-2))(b-a(n-1))},$$

where

$$\begin{aligned} K_4(a, b, c, d) &= -2a^2, \\ K_3(a, b, c, d) &= 4a(3a+2b), \\ K_2(a, b, c, d) &= -8(3a^2+a(4b+c)+b^2), \\ K_1(a, b, c, d) &= 2(3a+2b)(3a+4(b+c)) - 2a(-1)^n(a+2b+4c+8d), \\ K_0(a, b, c, d) &= ((-1)^n-1)(3a+2b)(a+2b+4c+8d). \end{aligned}$$

For $n = 2m$ and $n = 2m+1$, γ_n in (21) becomes

$$\gamma_{2m} = \frac{m(-a^2(m-1)^3 + a(2b(m-1)^2 + c(1-m) - d) + b(b(1-m) + c))}{(b-2a(m-1))(b-a(2m-1))},$$

and

$$\gamma_{2m+1} = \frac{(a(m-1)-b)(-am^3+bm^2-cm+d)}{(2am-(a+b))(2am-b)}.$$

Remark 1. Once we have explicitly determined γ_n in the recurrence relation (15), a discussion about the situation of this coefficient is extremely important. For instance, analyzing the location of the zeros of orthogonal polynomials would give rise to a positive definite case when $\gamma_n > 0$ ($\forall n \in \mathbf{N}$), the quasi-definite case when $\gamma_n \neq 0$, and weak orthogonality case when $\gamma_n = 0$ for some values of n . However, this discussion completely depends on the four parameters a, b, c and d , because γ_n is in general a rational expression in n . In section 4 we provide factorized representations of γ_n in two hypergeometric sequences, which would allow us to easily determine the orthogonality situation based on the parameters.

Theorem 3. The explicit form of the polynomial $S_n(a, b, c, d; x)$ is

$$(22) \quad S_n \left(\begin{array}{cc|c} a & b & x \\ c & d & \end{array} \right) = x^{\sigma_n} \sum_{j=0}^{[n/2]} (-1)^j \binom{[n/2]}{j} \\ \times \left(\prod_{i=j}^{[n/2]-1} \frac{a(i+\sigma_n)^3 - b(i+\sigma_n)^2 + c(i+\sigma_n) - d}{a(i+[n/2] - \sigma_{n+1}) - b} \right) (\sigma_n - x)_j (\sigma_n + x)_j,$$

where $[x]$ denotes the integer part of x , $(A)_n = A(A+1) \cdots (A+n-1) = \Gamma(A+n)/\Gamma(A)$ for $n \geq 1$ with $(A)_0 = 1$ and $\prod_{i=0}^{-1} (\cdot) = 1$.

Moreover, since $(-x)_j (x)_j = (-1)^j \prod_{k=0}^{j-1} (x^2 - k^2)$, for $n = 2m$ and $n = 2m+1$ (22) becomes

$$S_{2m} \left(\begin{array}{cc|c} a & b & x \\ c & d & \end{array} \right) = P_m(x^2) = \sum_{j=0}^m \binom{m}{j} \left(\prod_{i=j}^{m-1} \frac{ai^3 - bi^2 + ci - d}{a(i+m-1) - b} \right) \prod_{k=0}^{j-1} (x^2 - k^2),$$

and

$$S_{2m+1} \left(\begin{array}{cc|c} a & b & x \\ c & d & \end{array} \right) = x R_m(x^2) \\ = x \sum_{j=0}^m \binom{m}{j} \left(\prod_{i=j}^{m-1} \frac{a(i+1)^3 - b(i+1)^2 + c(i+1) - d}{a(i+m) - b} \right) \prod_{k=1}^j (x^2 - k^2).$$

Proof. By defining the symmetric basis

$$(23) \quad \vartheta_n(x) = (-1)^{[n/2]} x^{\sigma_n} (\sigma_n - x)_{[n/2]} (\sigma_n + x)_{[n/2]} \\ = (-1)^{[n/2]} x^{\sigma_n} \prod_{k=0}^{[n/2]-1} ((k + \sigma_n)^2 - x^2) = (-1)^n \vartheta_n(-x),$$

the following straightforward properties are derived

$$(24) \quad \Delta \vartheta_n(x) = n \vartheta_{n-1}(x) + \frac{n(n-1)}{2} \vartheta_{n-2}(x) + \sigma_{n+1} \frac{n(n-1)(n-2)}{4} \vartheta_{n-3}(x),$$

$$(25) \quad x \vartheta_n(x) = \vartheta_{n+1}(x) + \sigma_{n+1} \left(\frac{n}{2} \right)^2 \vartheta_{n-1}(x),$$

$$(26) \quad \Delta \nabla \vartheta_n(x) = n(n-1) \vartheta_{n-2}(x).$$

The expansion of $S_n(a, b, c, d; x)$ in terms of the above basis yields

$$(27) \quad S_n(a, b, c, d; x) = \sum_{j=0}^{[n/2]} c_j(n) \vartheta_{2j+\sigma_n}(x) \\ = \begin{cases} \sum_{j=0}^m (-1)^j c_j(2m) (-x)_j (x)_j, & n = 2m, \\ \sum_{j=0}^m (-1)^j c_j(2m+1) x(1-x)_j (1+x)_j, & n = 2m+1, \end{cases}$$

then by using the Navima algorithm [2, 30, 31] we can reach a solvable recurrence relation for the connection coefficients $c_j(n)$. So, by using the latter properties (24)–(26) and substituting (27) in (20) we obtain

$$0 = \sum_{j=0}^{[n/2]} c_j(n) (E_j(n) + \vartheta_{2j+\sigma_n+2} + F_j(n) \vartheta_{2j+\sigma_n} + G_j(n) \vartheta_{2j+\sigma_n-2}),$$

where

$$E_j(n) = 2(2j + \sigma_n - n)(a(2j + n - 2) + a\sigma_n - 2b),$$

$$F_j(n) = 4a(j-1)^2 j(2j-1)\sigma_{n-1} \\ + \frac{1}{2}\sigma_{n+2}(2j + \sigma_n + 1)^2(2j + \sigma_n - n)(a(2j + n - 2) + a\sigma_n - 2b) \\ + 2j\sigma_{n+1}(a(4j^3 - 6j^2 - j(n-3)(n+1) - 1) + 2b(j(-4j + n + 3) - 1)) \\ + \frac{1}{2}(\sigma_n(\sigma_n(\sigma_n(8aj + a\sigma_n - 4a - 2b) + a(24(j-1)j + 5) + 4b(1-3j)) \\ + 2a(2j-1)(8(j-1)j + 1) + 8b(2-3j)j + 4c) + a(4(j-1)j(1-2j)^2 + 1) \\ + 4j(b(1-4(j-1)j) + 4c) - 4c) - 2n(a+b) + an^2 \\ - 4j(b+4(c+d)) + 8j^2(b+2c)),$$

and

$$G_j(n) = (j-1)j(2j-1)(2(j-1)\sigma_{n-1}(2a(j-1)^2\sigma_{n-1} + b + 2c) \\ - 2\sigma_{n+1}((j-1)^2(a+2b)\sigma_{n-1} + c + 2d)) \\ + \frac{1}{8}(2j + \sigma_n - 1)(2j + \sigma_n)(\sigma_n(2j + \sigma_n - 1)(\sigma_n(2((j-1)(-4j(a+b) \\ + 4aj^2 + a) + 2c) + \sigma_n(\sigma_n(6aj + a\sigma_n - 4a - 2b) \\ + (2j-1)(a(6j-5) - 4b))) + b(4j-2) - 8(c+d) + 8cj) + 8d).$$

But since $\vartheta_n(x)$ is linearly independent, the coefficients $c_j(n)$ would satisfy the relation

$$E_{j-1}(n)c_{j-1}(n) + F_j(n)c_j(n) + G_{j+1}(n)c_{j+1}(n) = 0,$$

which is explicitly solvable with the initial conditions $c_m(n) = 0$ for $m > [n/2]$, and $c_{[n/2]}(n) = 1$, providing (22). \square

Remark 2. By substituting the characteristic vector

$$(a, b, c, d) = \left(\frac{p^*}{q^*}h, -\frac{p^* + r^*}{2q^*}h, \frac{1}{h}, -\frac{q^* + s^*}{2q^*h} \right),$$

and $x = t/h$ in relation (22), the corresponding polynomial satisfies a difference equation of type (20) that formally tends to the differential equation (12) as $h \rightarrow 0$. Hence, the following limit relation can be directly deduced:

$$\lim_{h \rightarrow 0} S_n \left(\begin{array}{c} \frac{p^*}{q^*}h \\ \frac{1}{h} \end{array} \middle| \begin{array}{c} -\frac{p^* + r^*}{2q^*}h \\ -\frac{q^* + s^*}{2q^*h} \end{array} \middle| \frac{t}{h} \right) = S_n^* \left(\begin{array}{c} r^* \\ p^* \end{array} \begin{array}{c} s^* \\ q^* \end{array} \middle| t \right).$$

As the recurrence relation (15) is explicitly known, the complete form of the orthogonality relation is

$$(28) \quad \sum_{x=-\theta}^{\theta} W \left(\begin{array}{c} a \\ c \end{array} \begin{array}{c} b \\ d \end{array} \middle| x \right) S_n \left(\begin{array}{c} a \\ c \end{array} \begin{array}{c} b \\ d \end{array} \middle| x \right) S_m \left(\begin{array}{c} a \\ c \end{array} \begin{array}{c} b \\ d \end{array} \middle| x \right) \\ = \prod_{k=1}^n \gamma_k \left(\begin{array}{c} a \\ c \end{array} \begin{array}{c} b \\ d \end{array} \right) \left(\sum_{x=-\theta}^{\theta} W \left(\begin{array}{c} a \\ c \end{array} \begin{array}{c} b \\ d \end{array} \middle| x \right) \right) \delta_{n,m},$$

where

$$W \left(\begin{array}{c} a \\ c \end{array} \begin{array}{c} b \\ d \end{array} \middle| x \right) = \left(\frac{1}{4} - x^2 \right) W(x),$$

is the original weight function and $W(x)$ satisfies the difference equation

$$(29) \quad \frac{W(x+1)}{W(x)} = \frac{(1/2) - x}{(3/2) + x} \frac{-ax^3 + bx^2 - cx + d}{a(x+1)^3 + b(x+1)^2 + c(x+1) + d}.$$

By noting that $A(x) = (2x+1)(ax^3 + bx^2 + cx + d)$ for $|a| + |b| \neq 0$, two cases can generally happen for the parameter a in (29), i.e. when $a \neq 0$ and b arbitrary or $a = 0$ and $b \neq 0$.

In the first case, since any arbitrary polynomial of degree 3 has at least one real root, say $x = p \in \mathbf{R}$, the aforementioned $A(x)$ can be decomposed in three different forms, i.e.

$$(30) \quad A(x) = (2x+1)(x-p)(ax^2 + ux + v) \\ = \begin{cases} (2x+1)(x-p)(ax^2 + ux + v), & (u^2 < 4av), \\ (2x+1)a(x-p)(x-q)(x-r), & (u^2 > 4av), \\ (2x+1)a(x-p)(x-q)^2, & (u^2 = 4av). \end{cases}$$

Similarly, in the second case when $a = 0$ and $b \neq 0$, $A(x)$ can be decomposed as

$$(31) \quad A(x) = (2x+1)(bx^2 + cx + d) \\ = \begin{cases} (2x+1)(bx^2 + cx + d), & (c^2 < 4bd), \\ (2x+1)b(x-p)(x-q), & (c^2 > 4bd), \\ (2x+1)b(x-p)^2, & (c^2 = 4bd). \end{cases}$$

For the two sub-cases $A(x) = (2x+1)(x-p)(ax^2 + ux + v)$ in (30) and $A(x) = (2x+1)(bx^2 + cx + d)$ in (31), the difference equations corresponding to (29)

respectively take the forms

$$(32) \quad \frac{W(x+1)}{W(x)} = \frac{(1/2) - x}{(3/2) + x} \frac{p+x}{p-x-1} \frac{ax^2 - ux + v}{a(x+1)^2 + u(x+1) + v} \quad (u^2 < 4av),$$

and

$$(33) \quad \frac{W(x+1)}{W(x)} = \frac{(1/2) - x}{(3/2) + x} \frac{bx^2 - cx + d}{b(x+1)^2 + c(x+1) + d} \quad (c^2 < 4bd).$$

Since the denominators of the two fractions (32) and (33) are not decomposable in \mathbf{R} , we shall deal with them in a separate work [under preparation]. However, it is important to note that the cases analyzed in this paper allow us to recover all classical symmetric orthogonal polynomials of a discrete variable (section 6). Hence, let us consider the rest of cases. For the second sub-case of (30), without loss of generality take $a = 1$ and then $A(x) = (2x+1)(x-p)(x-q)(x-r)$ where $p, q, r \in \mathbf{R}$, which indeed covers the third sub-case too. For the second sub-case of (31) we can similarly consider $A(x) = (2x+1)(x-p)(x-q)$ where $p, q \in \mathbf{R}$. This means that there exist only two orthogonal sequences of $S_n(a, b, c, d; x)$ when $A(x)$ is decomposable. Moreover, there exist 20 different weight functions on 32 supports for these two sequences, as the weight functions are not in general unique leading to an indeterminate moment problem. In summary, when the polynomial $A(x)$ is of degree four, a three parametric family (with 14 subcases) appears and when $A(x)$ is of degree three, a two parametric family (with 6 subcases) would appear. Finally, when $A(x)$ is of degree 2, the well known classical symmetric discrete families appear.

3. ORTHOGONALITY SUPPORTS

The question is now how to determine the restrictions of the parameter θ in the orthogonality support $[-\theta, \theta]$ in (28)? To answer, we should reconsider the main difference equation (20) on $[\alpha, \beta - 1] = [-\theta, \theta]$ and write it in a self-adjoint form to eventually obtain

$$(34) \quad \sum_{x=-\theta}^{\theta} \Delta (A(x)W(x) (\phi_m(x)\nabla\phi_n(x) - \phi_n(x)\nabla\phi_m(x))) \\ + (\lambda_n - \lambda_m) \sum_{x=-\theta}^{\theta} \left(\frac{1}{4} - x^2 \right) W(x)\phi_n(x)\phi_m(x) \\ + \frac{(-1)^m - (-1)^n}{2} \left(\frac{a}{2} + b + 2c + 4d \right) \sum_{x=-\theta}^{\theta} W(x)\phi_n(x)\phi_m(x) = 0.$$

On the other side, the identity

$$\phi_m(x)\nabla\phi_n(x) - \phi_n(x)\nabla\phi_m(x) = \phi_n(x)\phi_m(x-1) - \phi_m(x)\phi_n(x-1),$$

simplifies the first sum of (34) as

$$\begin{aligned}
 (35) \quad & \sum_{x=-\theta}^{\theta} \Delta (A(x)W(x) (\phi_m(x)\nabla\phi_n(x) - \phi_n(x)\nabla\phi_m(x))) \\
 &= A(x)W(x) \left(\phi_n(x)\phi_m(x-1) - \phi_m(x)\phi_n(x-1) \right) \Big|_{x=-\theta}^{x=\theta+1} \\
 &= A(\theta+1)W(\theta+1) (\phi_n(\theta+1)\phi_m(\theta) - \phi_m(\theta+1)\phi_n(\theta)) \\
 &\quad - A(-\theta)W(-\theta) (\phi_n(-\theta)\phi_m(-\theta-1) - \phi_m(-\theta)\phi_n(-\theta-1)).
 \end{aligned}$$

By taking into account that all weight functions are even, i.e. $W(-x) = W(x)$, the polynomials are symmetric, i.e. $\phi_n(x) = (-1)^n \phi_n(-x)$, and the Pearson difference equation (11) is also valid for $x = \theta$, i.e. $A(\theta+1)W(\theta+1) = A(-\theta)W(\theta)$, relation (35) would be finally simplified as

$$\begin{aligned}
 (36) \quad & \sum_{x=-\theta}^{\theta} \Delta (A(x)W(x) (\phi_m(x)\nabla\phi_n(x) - \phi_n(x)\nabla\phi_m(x))) \\
 &= A(-\theta)W(\theta) (1 + (-1)^{n+m}) (\phi_m(\theta)\phi_n(\theta+1) - \phi_n(\theta)\phi_m(\theta+1)).
 \end{aligned}$$

Since $\phi_m(\theta)\phi_n(\theta+1) - \phi_n(\theta)\phi_m(\theta+1) \neq 0$, two cases can in general happen for the right hand side of (36):

- i) If $n + m$ is odd then $1 + (-1)^{n+m} = 0$ and (36) is automatically zero. However, this case is clear as the sum of any odd summand on a symmetric counter set is equal to zero.
- ii) If simultaneously $A(-\theta) = 0$ and $W(\theta) \neq 0$, then (36) is again equal to zero. This condition is in fact the key point for finding the orthogonality supports.

4. TWO HYPERGEOMETRIC ORTHOGONAL SEQUENCES OF $S_n(a, b, c, d; x)$

In this section, we introduce two hypergeometric sequences of symmetric orthogonal polynomials, which are particular cases of $S_n(a, b, c, d; x)$ corresponding to two aforementioned cases $a \neq 0$ and $a = 0$, respectively and then obtain all possible weight functions together with orthogonality supports for these two sequences.

4.1. First sequence. If the characteristic vector

$$(a, b, c, d) = (1, -(p+q+r), pq+pr+qr, -pqr),$$

for $p, q, r \in \mathbf{R}$, is replaced in (20), then

$$\begin{aligned}
 & (2x+1)(x-p)(x-q)(x-r)\Delta\nabla\phi_n(x) \\
 & - 2x(x^2(1-2p-2q-2r) + pq+pr+qr-2pqr)\Delta\phi_n(x) \\
 & + \left(2n(n+2(p+q+r-1)) \left(\frac{1}{4} - x^2 \right) \right. \\
 & \quad \left. - \frac{\sigma_n}{2}(2p-1)(2q-1)(2r-1) \right) \phi_n(x) = 0,
 \end{aligned}$$

has a polynomial solution, which can be represented in terms of hypergeometric series as

$$(37) \quad S_n \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \middle| x \right) = \frac{(p+\sigma_n)_{[n/2]}(q+\sigma_n)_{[n/2]}(r+\sigma_n)_{[n/2]}}{([n/2]+p+q+r-1+\sigma_n)_{[n/2]}} \\ \times x^{\sigma_n} {}_4F_3 \left(\begin{matrix} -[n/2], [n/2]+p+q+r-1+\sigma_n, \sigma_n-x, \sigma_n+x \\ p+\sigma_n, q+\sigma_n, r+\sigma_n \end{matrix} \middle| 1 \right),$$

where σ_n is defined in (6) and ${}_4F_3$ is the well known hypergeometric function of order (4, 3). Moreover, it satisfies the recurrence relation

$$(38) \quad \phi_{n+1}(x) = x\phi_n(x) - \gamma_n \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \right) \phi_{n-1}(x), \\ (\phi_0(x) = 1, \quad \phi_1(x) = x),$$

where

$$(39) \quad \gamma_n \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \right) = \{-2n^4 - 4(-3+2p+2q+2r)n^3 \\ -8(p^2+p(3q+3r-4)+3qr+(q-4)q+r^2-4r+3)n^2 \\ + (2(-1)^n(-1+2p)(-1+2q)(-1+2r) \\ -2(-3+2p+2q+2r)(3+4q(-1+r)-4r+4p(-1+q+r)))n \\ + (-1+(-1)^n)(-1+2p)(-1+2q)(-1+2r)(-3+2p+2q+2r)\} \\ / \{32(n+p+q+r-2)(n+p+q+r-1)\},$$

dividing to

$$\gamma_{2n} = -\frac{n(n+p+q-1)(n+p+r-1)(n+q+r-1)}{(2n+p+q+r-2)(2n+p+q+r-1)},$$

and

$$\gamma_{2n+1} = -\frac{(n+p)(n+q)(n+r)(n+p+q+r-1)}{(2n+p+q+r-1)(2n+p+q+r)}.$$

The latter representations would allow us to analyze the sign of γ_n in terms of the values of p, q and r as explained in Remark 1.

By noting the relations (38) and (39), the orthogonality relation of the first sequence is

$$\sum_{x=-\theta}^{\theta} \left(W \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \middle| x \right) S_n(x) S_m(x) \right) \\ = \prod_{k=1}^n \gamma_k \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \right) \\ \times \left(\sum_{x=-\theta}^{\theta} W \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \middle| x \right) \right) \delta_{n,m},$$

in which

$$S_n(x) = S_n \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \middle| x \right),$$

and, finally

$$W \left(\begin{matrix} 1 & -(p+q+r) \\ pq+pr+qr & -pqr \end{matrix} \middle| x \right) = \left(\frac{1}{4} - x^2 \right) W(x),$$

denotes the original weight function corresponding to the hypergeometric polynomial (37). As a consequence, we obtain

Proposition 2. *The function $W(x)$ satisfies a particular case of the difference equation (29)*

$$(40) \quad \frac{W(x+1)}{W(x)} = \frac{-x+(1/2)}{x+(3/2)} \frac{-x-p}{x+1-p} \frac{-x-q}{x+1-q} \frac{-x-r}{x+1-r},$$

giving rise to the 16 symmetric solutions of (40) listed below

$$(41) \quad W_1(x; p, q, r) = (\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+x)\Gamma(1-q-x) \\ \times \Gamma(1-r+x)\Gamma(1-r-x)\Gamma(3/2+x)\Gamma(3/2-x))^{-1},$$

$$(42) \quad W_{2,1}(x; p, q, r) \\ = \frac{\Gamma(p+x)\Gamma(p-x)}{\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(1-r+x)\Gamma(1-r-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(43) \quad W_{2,2}(x; p, q, r) = W_{2,1}(x; q, p, r) \\ = \frac{\Gamma(q+x)\Gamma(q-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-r+x)\Gamma(1-r-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(44) \quad W_{2,3}(x; p, q, r) = W_{2,1}(x; r, q, p) \\ = \frac{\Gamma(r+x)\Gamma(r-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(45) \quad W_{3,1}(x; p, q, r) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x)}{\Gamma(1-r+x)\Gamma(1-r-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(46) \quad W_{3,2}(x; p, q, r) = W_{3,1}(x; p, r, q) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(r+x)\Gamma(r-x)}{\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(47) \quad W_{3,3}(x; p, q, r) = W_{3,2}(x; r, q, p) = \frac{\Gamma(q+x)\Gamma(q-x)\Gamma(r+x)\Gamma(r-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(48) \quad W_{4,1}(x; p, q, r) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(1-r+x)\Gamma(1-r-x)},$$

$$(49) \quad W_{4,2}(x; p, q, r) = W_{4,1}(x; q, p, r) \\ = \frac{\Gamma(q+x)\Gamma(q-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-r+x)\Gamma(1-r-x)},$$

$$(50) \quad W_{4,3}(x; p, q, r) = W_{4,1}(x; r, q, p) \\ = \frac{\Gamma(r+x)\Gamma(r-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+x)\Gamma(1-q-x)},$$

$$(51) \quad W_{5,1}(x; p, q, r) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-r+x)\Gamma(1-r-x)},$$

$$(52) \quad W_{5,2}(x; p, q, r) = W_{5,1}(x; p, r, q) \\ = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(r+x)\Gamma(r-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-q+x)\Gamma(1-q-x)},$$

$$(53) \quad W_{5,3}(x; p, q, r) = W_{5,1}(x; r, q, p) \\ = \frac{\Gamma(q+x)\Gamma(q-x)\Gamma(r+x)\Gamma(r-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)},$$

$$(54) \quad W_6(x; p, q, r) \\ = \frac{\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(1-r+x)\Gamma(1-r-x)},$$

$$(55) \quad W_7(x; p, q, r) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x)\Gamma(r+x)\Gamma(r-x)}{\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(56) \quad W_8(x; p, q, r) = \Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x) \\ \times \Gamma(r+x)\Gamma(r-x)\Gamma(-1/2+x)\Gamma(-1/2-x).$$

As the original weight functions corresponding to all above 16 cases are as

$$\left(\frac{1}{4} - x^2\right) W_k(x),$$

the two following identities are remarkable in this direction

$$(1/4 - x^2)\Gamma(-1/2+x)\Gamma(-1/2-x) = \Gamma(1/2+x)\Gamma(1/2-x),$$

and

$$\frac{1/4 - x^2}{\Gamma(3/2+x)\Gamma(3/2-x)} = \frac{1}{\Gamma(1/2+x)\Gamma(1/2-x)}.$$

For example, for the last given case the original weight function becomes

$$W_8 \left(\begin{array}{cc|c} 1 & -(p+q+r) & x \\ pq+pr+qr & -pqr & \end{array} \right) = \left(\frac{1}{4} - x^2\right) W_8(x; p, q, r) \\ = \Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x)\Gamma(r+x)\Gamma(r-x)\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right).$$

By noting section 3, the following table now shows the orthogonality supports of each weight function given in (41) to (56) together with their parameter restrictions in which $\mathbf{Z}^- = \{0, -1, -2, \dots\}$.

TABLE 1. Orthogonality supports for the first hypergeometric sequence

$W_k(x)$	Support	Parameter restrictions
$W_1(x; p, q, r)$	$[-p, p]$ $[-q, q]$ $[-r, r]$	$p \in \mathbf{Z}^-, 1 - q \pm p \notin \mathbf{Z}^-, 1 - r \pm p \notin \mathbf{Z}^-.$ $q \in \mathbf{Z}^-, 1 - p \pm q \notin \mathbf{Z}^-, 1 - r \pm q \notin \mathbf{Z}^-.$ $r \in \mathbf{Z}^-, 1 - q \pm r \notin \mathbf{Z}^-, 1 - p \pm r \notin \mathbf{Z}^-.$
$W_{2,1}(x; p, q, r)$	$[-q, q]$ $[-r, r]$	$q \in \mathbf{Z}^-, p \pm q \notin \mathbf{Z}^-, 1 - r \pm q \notin \mathbf{Z}^-.$ $r \in \mathbf{Z}^-, p \pm r \notin \mathbf{Z}^-, 1 - q \pm r \notin \mathbf{Z}^-.$
$W_{3,1}(x; p, q, r)$	$[-r, r]$	$r \in \mathbf{Z}^-, p \pm r \notin \mathbf{Z}^-, q \pm r \notin \mathbf{Z}^-.$
$W_{4,1}(x; p, q, r)$	$[-q, q]$ $[-r, r]$	$q \in \mathbf{Z}^-, p \pm q \notin \mathbf{Z}^-, 1 - r \pm q \notin \mathbf{Z}^-.$ $r \in \mathbf{Z}^-, p \pm r \notin \mathbf{Z}^-, 1 - q \pm r \notin \mathbf{Z}^-.$
$W_{5,1}(x; p, q, r)$	$[-r, r]$	$r \in \mathbf{Z}^-, p \pm r \notin \mathbf{Z}^-, q \pm r \notin \mathbf{Z}^-.$
$W_6(x; p, q, r)$	$[-p, p]$ $[-q, q]$ $[-r, r]$	$p \in \mathbf{Z}^-, 1 - q \pm p \notin \mathbf{Z}^-, 1 - r \pm q \notin \mathbf{Z}^-.$ $q \in \mathbf{Z}^-, 1 - r \pm q \notin \mathbf{Z}^-, 1 - p \pm q \notin \mathbf{Z}^-.$ $r \in \mathbf{Z}^-, 1 - q \pm r \notin \mathbf{Z}^-, 1 - p \pm r \notin \mathbf{Z}^-.$
$W_7(x; p, q, r)$	—	—
$W_8(x; p, q, r)$	—	—

Remark 3. Since some weight functions are symmetric with respect to the parameters p , q and r , e.g. $W_{4,2}(x; p, q, r) = W_{4,1}(x; q, p, r)$ and $W_{4,3}(x; p, q, r) = W_{4,1}(x; r, q, p)$, their orthogonality supports and parameter restrictions can be directly derived via the rows of table 1 by just interchanging the parameters. Also note that $W_7(x; p, q, r)$ and $W_8(x; p, q, r)$ have no valid orthogonality support. Therefore, there are totally 24 eligible orthogonality supports for the first sequence.

Remark 4. Let the weight functions corresponding to the first sequence be indicated as $\varrho_i(x) = (1/4 - x^2)W_i(x)$. Then they satisfy the difference equation

$$\frac{\varrho_i(x+1)}{\varrho_i(x)} = \frac{(p+x)(q+x)(r+x)}{(-p+x+1)(-q+x+1)(-r+x+1)},$$

which is equivalent to

$$\Delta(M_1(x)\varrho_i(x)) = N_1(x)\varrho_i(x),$$

where

$$M_1(x) = (x-p)(x-q)(x-r), \quad N_1(x) = 2x^2(p+q+r) + 2pqr.$$

Thus, the hypergeometric polynomials (37) constitute a Δ -semiclassical sequence of class one [18], for which we have explicitly given their orthogonality condition, three term recurrence relation, hypergeometric representation and in the sequel in section 5 we will compute the factorial moments with respect to the basis $\vartheta_n(x)$ defined in (23).

4.1.1. *A numerical example for the first sequence.* Let us replace $p = -9$, $q = -10$ and $r = -11$ in (37) to get

$$(57) \quad S_n \left(\begin{array}{cc|c} 1 & 30 & x \\ 299 & 990 & \end{array} \right) = \frac{(-9 + \sigma_n)_{[n/2]}(-10 + \sigma_n)_{[n/2]}(-11 + \sigma_n)_{[n/2]}}{(-31 + [n/2] + \sigma_n)_{[n/2]}} \times x^{\sigma_n} {}_4F_3 \left(\begin{array}{c} -[n/2], -31 + [n/2] + \sigma_n, \sigma_n - x, \sigma_n + x \\ -9 + \sigma_n, -10 + \sigma_n, -11 + \sigma_n \end{array} \middle| 1 \right).$$

Since $p \in \mathbf{Z}^-$, $1 - q \pm p \notin \mathbf{Z}^-$ and $1 - r \pm p \notin \mathbf{Z}^-$, referring to Table 1 shows that there are two possible weight functions for the selected parameters which are orthogonal with respect to the sequence $S_n(1, 30, 299, 990; x)$ on the support $\{-9, -8, \dots, 8, 9\}$, i.e.

$$\begin{aligned} W_1 \left(\begin{array}{cc|c} 1 & 30 & x \\ 299 & 990 & \end{array} \right) &= \left(\frac{1}{4} - x^2 \right) W_1(x; -9, -10, -11) \\ &= (\Gamma(10 + x)\Gamma(10 - x)\Gamma(11 + x)\Gamma(11 - x) \\ &\quad \Gamma(12 + x)\Gamma(12 - x)\Gamma(1/2 + x)\Gamma(1/2 - x))^{-1}, \end{aligned}$$

and

$$\begin{aligned} W_6 \left(\begin{array}{cc|c} 1 & 30 & x \\ 299 & 990 & \end{array} \right) &= \left(\frac{1}{4} - x^2 \right) W_6(x; -9, -10, -11) \\ &= \frac{\Gamma(1/2 + x)\Gamma(1/2 - x)}{\Gamma(10 + x)\Gamma(10 - x)\Gamma(11 + x)\Gamma(11 - x)\Gamma(12 + x)\Gamma(12 - x)}. \end{aligned}$$

Hence there are two orthogonality relations corresponding to the polynomials (57), respectively as follows

$$\begin{aligned} &\sum_{x=-9}^9 W_i \left(\begin{array}{cc|c} 1 & 30 & x \\ 299 & 990 & \end{array} \right) S_n(1, 30, 299, 990; x) S_m(1, 30, 299, 990; x) \\ &= \alpha_i \frac{36183421612800000(-1)^n(-29)_{\lfloor \frac{n-1}{2} \rfloor - 1}(-19)_{\lfloor \frac{n}{2} \rfloor - 2}(-18)_{\lfloor \frac{n}{2} \rfloor - 2}(-17)_{\lfloor \frac{n}{2} \rfloor - 2}}{(-31)_n(-30)_n} \\ &\quad \times (-9)_{\lfloor \frac{n-1}{2} \rfloor - 1}(-8)_{\lfloor \frac{n-1}{2} \rfloor - 1}(-7)_{\lfloor \frac{n-1}{2} \rfloor - 1} \Gamma \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \delta_{n,m}, \end{aligned}$$

where $\lfloor x \rfloor$ denotes the floor function, and

$$\alpha_i = \sum_{x=-9}^9 W_i \left(\begin{array}{cc|c} 1 & 30 & x \\ 299 & 990 & \end{array} \right) = \begin{cases} \beta/\pi, & i = 1, \\ \beta\pi, & i = 6, \end{cases}$$

in which

$$\beta = \frac{667}{1998530094928466929986605067918114816000000000}.$$

4.2. **Second sequence.** If the characteristic vector

$$(a, b, c, d) = (0, 1, -p - q, pq),$$

for $p, q \in \mathbf{R}$, is replaced in the difference equation (20), then

$$(2x+1)(x-p)(x-q)\Delta\nabla\phi_n(x) - 2x(2x^2+p(2q-1)-q)\Delta\phi_n(x) \\ + \left(-4n\left(\frac{1}{4}-x^2\right) + \frac{1-(-1)^n}{2}(2p-1)(2q-1)\right)\phi_n(x) = 0,$$

has a basis solution, which can be written in terms of hypergeometric series

$$(58) \quad S_n \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \middle| x \right) = (p+\sigma_n)_{[n/2]}(q+\sigma_n)_{[n/2]} \\ \times x^{\sigma_n} {}_3F_2 \left(\begin{array}{c} -[n/2], \sigma_n - x, \sigma_n + x \\ p + \sigma_n, q + \sigma_n \end{array} \middle| 1 \right).$$

The above polynomial is a limit case of the polynomial (37) when $r \rightarrow \infty$. Moreover, it satisfies a recurrence relation of type (15) with

$$\gamma_n \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \right) = \frac{1}{8} (-2n^2 - 4n(p+q-1) + ((-1)^n - 1)(2p-1)(2q-1)),$$

which implies

$$\gamma_{2n} = -n(-1+n+p+q),$$

and

$$\gamma_{2n+1} = -(n+p)(n+q).$$

Hence, the orthogonality relation corresponding to the second sequence takes the form

$$\sum_{x=-\theta}^{\theta} \left(W \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \middle| x \right) \right. \\ \times S_n \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \middle| x \right) S_m \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \middle| x \right) \Bigg) \\ = \prod_{k=1}^n \gamma_k \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \right) \left(\sum_{x=-\theta}^{\theta} W \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \middle| x \right) \right) \delta_{n,m},$$

where

$$W \left(\begin{array}{cc} 0 & 1 \\ -p-q & pq \end{array} \middle| x \right) = \left(\frac{1}{4} - x^2 \right) W(x),$$

is the original weight function. Thus, we obtain

Proposition 3. *The function $W(x)$ satisfies a particular case of the difference equation (29) as*

$$(59) \quad \frac{W(x+1)}{W(x)} = \frac{1/2-x}{3/2+x} \frac{-x-p}{x+1-p} \frac{-x-q}{x+1-q},$$

giving rise to 8 symmetric solutions for equation (59) respectively as follows:

$$(60) \quad W_9(x; p, q) \\ = \frac{1}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(61) \quad W_{10,1}(x; p, q) = \frac{\Gamma(p+x)\Gamma(p-x)}{\Gamma(1-q+x)\Gamma(1-q-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(62) \quad W_{10,2}(x; p, q) = \frac{\Gamma(q+x)\Gamma(q-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(63) \quad W_{11}(x; p, q) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x)}{\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$(64) \quad W_{12}(x; p, q) = \frac{\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+x)\Gamma(1-q-x)},$$

$$(65) \quad W_{13,1}(x; p, q) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-q+x)\Gamma(1-q-x)},$$

$$(66) \quad W_{13,2}(x; p, q) = \frac{\Gamma(q+x)\Gamma(q-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)},$$

$$(67) \quad W_{14}(x; p, q) = \Gamma(p+x)\Gamma(p-x)\Gamma(q+x)\Gamma(q-x)\Gamma(-1/2+x)\Gamma(-1/2-x).$$

Table 2 shows the orthogonality supports of each above-mentioned weights and their parameter restrictions.

TABLE 2. Orthogonality supports for the second hypergeometric sequence

$W_k(x)$	Support	Parameter restrictions
$W_9(x; p, q)$	$[-p, p]$ $[-q, q]$	$p \in \mathbf{Z}^-, 1 - q \pm p \notin \mathbf{Z}^-.$ $q \in \mathbf{Z}^-, 1 - p \pm q \notin \mathbf{Z}^-.$
$W_{10,1}(x; p, q)$	$[-q, q]$	$q \in \mathbf{Z}^-, p \pm q \notin \mathbf{Z}^-.$
$W_{11}(x; p, q)$	—	—
$W_{12}(x; p, q)$	$[-p, p]$ $[-q, q]$	$p \in \mathbf{Z}^-, 1 - q \pm p \notin \mathbf{Z}^-.$ $q \in \mathbf{Z}^-, 1 - p \pm q \notin \mathbf{Z}^-.$
$W_{13,1}(x; p, q)$	$[-q, q]$	$q \in \mathbf{Z}^-, p \pm q \notin \mathbf{Z}^-.$
$W_{14}(x; p, q)$	—	—

Remark 5. As the main orthogonality relation (28) shows, an important part that one has to compute in norm square value is $\sum_{x=-\theta}^{\theta} W(x; a, b, c, d)$ where $[-\theta, \theta]$ is the same orthogonality supports as determined in tables 1 and 2. Since the functions $\{W_k(x; a, b, c, d)\}_{k=1}^{20}$ are all even, the aforesaid sums can be simplified on its orthogonality supports by using two identities

$$\Gamma(p+x) = \Gamma(p)(p)_x \quad \text{and} \quad \Gamma(p-x) = \frac{\Gamma(p)(-1)^x}{(-p)_x},$$

and this fact that

$$\sum_{x=-\theta}^{\theta} W(x; a, b, c, d) = 2 \sum_{x=0}^{\theta} W(x; a, b, c, d) - W(0; a, b, c, d).$$

For example, for the first given weight function $W_1(x; p, q, r)$ on e.g. $[-p, p]$ we have

$$\begin{aligned} \sum_{x=-p}^p \left(\frac{1}{4} - x^2 \right) W_1(x; p, q, r) \\ = \frac{1}{\pi \Gamma^2(1-p) \Gamma^2(1-q) \Gamma^2(1-r)} \left(2 \sum_{x=0}^p \frac{(p)_x (q)_x (r)_x}{(1-p)_x (1-q)_x (1-r)_x} - 1 \right). \end{aligned}$$

Remark 6. Let the weight functions corresponding to the second sequence be indicated as $\varrho_j(x) = (1/4 - x^2)W_j(x)$. Then they satisfy the difference equation

$$\frac{\varrho_j(x+1)}{\varrho_j(x)} = -\frac{(p+x)(q+x)}{(-p+x+1)(-q+x+1)},$$

which is equivalent to

$$\Delta(M_2 \varrho_j(x)) = N_2 \varrho_j(x),$$

where

$$M_2(x) = (x-p)(x-q), \quad N_2(x) = -2(pq + x^2).$$

4.2.1. *A numerical example for the second sequence.* Let us replace $p = 14$ and $q = -10$ in (37) to get

$$\begin{aligned} (68) \quad S_n \left(\begin{array}{cc|c} 0 & 1 & x \\ -4 & -140 & \end{array} \right) &= (14 + \sigma_n)_{[n/2]} (-10 + \sigma_n)_{[n/2]} \\ &\quad \times x^{\sigma_n} {}_3F_2 \left(\begin{array}{c|c} -[n/2], \sigma_n - x, \sigma_n + x & 1 \\ 14 + \sigma_n, -10 + \sigma_n & \end{array} \right). \end{aligned}$$

Since $q \in \mathbf{Z}^-$ and $p \pm q \notin \mathbf{Z}^-$, referring to Table 2 shows that there are two possible weight functions for the selected parameters which are orthogonal with respect to the sequence $\S_n(0, 1, -4, -140; x)$ on the support $\{-10, -9, \dots, 9, 10\}$, i.e.

$$\begin{aligned} W_{10,1} \left(\begin{array}{cc|c} 0 & 1 & x \\ -4 & -140 & \end{array} \right) &= \left(\frac{1}{4} - x^2 \right) W_{10,1}(x; 14, -10) \\ &= \frac{(13-x)(12-x)(11-x)(x+11)(x+12)(x+13) \cos(\pi x)}{\pi}, \end{aligned}$$

and

$$\begin{aligned} W_{13,1} \left(\begin{array}{cc|c} 0 & 1 & x \\ -4 & -140 & \end{array} \right) &= \left(\frac{1}{4} - x^2 \right) W_{13,1}(x; 14, -10) \\ &= \pi(13-x)(12-x)(11-x)(x+11)(x+12)(x+13) \sec(\pi x). \end{aligned}$$

Hence there are two orthogonality relations corresponding to the polynomials (68) respectively as follows

$$\begin{aligned} \sum_{x=-10}^{10} W_{j,1} \left(\begin{array}{cc|c} 0 & 1 & x \\ -4 & -140 & \end{array} \right) S_n(0, 1, -4, -140; x) S_m(0, 1, -4, -140; x) \\ = \tilde{\alpha}_j \frac{(-1)^{n+1}(-9) \lfloor \frac{n-1}{2} \rfloor \Gamma \left(\lfloor \frac{n-1}{2} \rfloor + 15 \right) \Gamma \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \Gamma \left(\lfloor \frac{n}{2} \rfloor + 4 \right)}{3736212480}, \end{aligned}$$

where

$$\tilde{\alpha}_j = \sum_{x=-10}^{10} W_{j,1} \left(\begin{array}{cc|c} 0 & 1 & x \\ -4 & -140 & \end{array} \right) = \begin{cases} 10296/\pi, & j = 10, \\ 10296\pi, & j = 13. \end{cases}$$

5. MOMENTS OF THE TWO INTRODUCED SEQUENCES

To compute the moments of a continuous distribution, different bases should be considered. For example, in the normal distribution the canonical basis $\{x^j\}_{j \geq 0}$ is used to get

$$\int_{-\infty}^{\infty} x^n \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \begin{cases} 0, & n = 2m + 1, \\ \frac{2^{m+1/2}}{\sqrt{2\pi}} \Gamma \left(m + \frac{1}{2} \right), & n = 2m, \end{cases}$$

while for the Jacobi weight function $(1-x)^\alpha(1+x)^\beta$ as the shifted beta distribution on $[-1, 1]$, using one of the two bases $\{(1-x)^j\}_j$ or $\{(1+x)^j\}_j$ is appropriate for this purpose.

This matter similarly holds for the moments of discrete orthogonal polynomials. For instance, in the negative hypergeometric distribution corresponding to Hahn polynomials, it is more convenient to use the Pochhammer basis $\{(-x)_n\}_{n \geq 0}$, instead of the canonical basis, to get

$$\begin{aligned} \sum_{x=0}^{N-1} \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+x+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(x+1)} (-x)_n \\ = (-1)^n \frac{(1-N)_n(\beta+1)_n}{(\alpha+\beta+2)_n}. \end{aligned}$$

Following this approach, for the weight functions $\varrho_i(x)$ appeared in the two introduced hypergeometric sequences, we shall compute the moments of the form

$$(\varrho_i)_n = \sum_{x=-\theta}^{\theta} \vartheta_n(x) \varrho_i(x) = \sum_{x=-\theta}^{\theta} \vartheta_n(x) \left(\frac{1}{4} - x^2 \right) W_i(x).$$

where the basis $\vartheta_n(x)$ is defined in (23).

Since $\vartheta_{2n+1}(x)$ are odd polynomials, clearly all odd moments with respect to this basis are zero. Moreover, from (37) and (58) and also using the orthogonality property of the polynomials it can be proved by induction that the even moments corresponding to the first and second sequences respectively satisfy the following recurrence relations

$$(\varrho_i)_{2n} = - \frac{(p+n-1)(q+n-1)(r+n-1)}{p+q+r+n-1} (\varrho_i)_{2n-2},$$

and

$$(\varrho_j)_{2n} = -(n+p-1)(n+q-1)(\varrho_j)_{2n-2}.$$

Therefore, if the aforesaid weight functions are normalized with the first moment equal to one, then we eventually obtain

$$\sum_{x=-\theta}^{\theta} \left(\frac{1}{4} - x^2 \right) W_i(x) \vartheta_n(x) = \begin{cases} 0, & n = 2m+1, \\ \frac{(-1)^m (p)_m (q)_m (r)_m}{(p+q+r)_m}, & n = 2m, \end{cases}$$

for the weight functions of the first sequence, and

$$\sum_{x=-\theta}^{\theta} \left(\frac{1}{4} - x^2 \right) W_j(x) \vartheta_n(x) = \begin{cases} 0, & n = 2m+1, \\ (-1)^m (p)_m (q)_m, & n = 2m. \end{cases}$$

for the weight functions of the second sequence.

6. A PARTICULAR EXAMPLE OF $S_n(a, b, c, d; x)$ GENERATING ALL CLASSICAL SYMMETRIC ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

In this section, we present a particular interesting example of $S_n(a, b, c, d; x)$ that generates all classical symmetric orthogonal polynomials of a discrete variable studied and analyzed in [3] and is different from the two introduced hypergeometric sequences.

If $a = -2(b+2c+4d)$ in the main difference equation (20), after simplification of a common factor we get

$$\begin{aligned} (x^2(b+2c+4d) + x(c+2d) + d) \Delta \nabla \phi_n(x) - 2x(c+2d) \Delta \phi_n(x) \\ + n(-(b(n-1) + 2(n-2)(c+2d))) \phi_n(x) = 0, \end{aligned}$$

which is the same equation as analyzed in [3],

$$(\hat{a}x^2 + \hat{b}x + \hat{c}) \Delta \nabla y_n(x) - 2\hat{b}x \Delta y(x) + n(\hat{a}(1-n) + 2\hat{b})y(x) = 0,$$

for

$$d = \hat{c}, \quad c = \hat{b} - 2\hat{c}, \quad b = \hat{a} - 2\hat{b}, \quad a = -2\hat{a}.$$

Hence, the particular polynomial

$$\begin{aligned} (69) \quad S_n \left(\begin{array}{cc|c} -2\hat{a} & \hat{a} - 2\hat{b} & \\ \hat{b} - 2\hat{c} & \hat{c} & \end{array} \middle| x \right) = x^{\sigma_n} \sum_{j=0}^{[n/2]} (-1)^j \binom{[n/2]}{j} \\ \times \left(\prod_{i=j}^{[n/2]-1} \frac{(2i+2\sigma_n+1) \left(-\hat{a}(i+\sigma_n)^2 + \hat{b}(i+\sigma_n) - \hat{c} \right)}{2 \left(\hat{b} + \hat{a}(\sigma_{n+1} - i - [n/2]) \right) - \hat{a}} \right) (\sigma_n - x)_j (\sigma_n + x)_j, \end{aligned}$$

generates all cases of classical symmetric orthogonal polynomials of a discrete variable respectively as follows:

Case 1. If $\hat{a} = 0$, $\hat{b} = 1$ and \hat{c} free in (69) then $\sigma(x) = (\hat{a}x^2 + \hat{b}x + \hat{c}) = x + \hat{c}$ has one real root and the symmetric Kravchuk polynomials [5, 12] are therefore

derived as

$$S_n \left(\begin{array}{cc|c} 0 & -2 & x \\ 1-2\hat{c} & \hat{c} & \end{array} \right) = 2^{-n} (-2\hat{c})_n {}_2F_1 \left(\begin{array}{c|c} -n, -\hat{c}-x & 2 \\ -2\hat{c} & \end{array} \right) \\ = k_n^{(1/2)}(x + \hat{c}; 2\hat{c}),$$

which are orthogonal with respect to the weight function

$$\varrho(x) = \frac{1}{\Gamma(\hat{c}-x+1)\Gamma(\hat{c}+x+1)} \quad \text{for } x \in \{-\hat{c}, -\hat{c}+1, \dots, \hat{c}-1, \hat{c}\},$$

when $2\hat{c} \in \mathbf{N}$.

Case 2. If $\hat{a} = 1$, \hat{b} free and $\hat{c} = \hat{b}^2/4$ in (69), then $\sigma(x) = (\hat{a}x^2 + \hat{b}x + \hat{c}) = (x + \hat{b}/2)^2$ has a double real root and the symmetric Hahn-Eberlein polynomials [28]

$$S_n \left(\begin{array}{cc|c} -2 & 1-2\hat{b} & x \\ \hat{b}(2-\hat{b})/2 & \hat{b}^2/4 & \end{array} \right) = \frac{((-b)_n)^2}{(n-2\hat{b}-1)_n} \\ \times {}_3F_2 \left(\begin{array}{c|c} -n, n-2\hat{b}-1, -x-\hat{b}/2 & 1 \\ -\hat{b}, -\hat{b} & \end{array} \right) = \tilde{h}_n^{(0,0)}(x + \hat{b}/2, \hat{b}+1),$$

are orthogonal with respect to the weight function

$$\varrho(x) = \frac{1}{\Gamma^2(x+1+\hat{b}/2)\Gamma^2(-x+1+\hat{b}/2)},$$

for $x \in \{-\hat{b}/2, -\hat{b}/2+1, \dots, \hat{b}/2-1, \hat{b}/2\}$ when $b \in \mathbf{N}$.

Case 3. If $\hat{a} = 1$, $\hat{b} = -\delta_1 - \delta_2$ and $\hat{c} = \delta_1\delta_2$ in (69), then $\sigma(x)$ has two different real roots δ_1 and δ_2 and depending on the values δ_1 and δ_2 , the symmetric Hahn-Eberlein polynomials [3, 28] are derived as

$$S_n \left(\begin{array}{cc|c} -2 & 1+2(\delta_1+\delta_2) & x \\ -\delta_2-\delta_1(1+2\delta_2) & \delta_1\delta_2 & \end{array} \right) \\ = \frac{(\delta_1+\delta_2)_n (2\delta_2)_n}{(n+2(\delta_1+\delta_2)-1)_n} {}_3F_2 \left(\begin{array}{c|c} -n, n+2(\delta_1+\delta_2)-1, \delta_2-x & 1 \\ \delta_1+\delta_2, 2\delta_2 & \end{array} \right) \\ = \tilde{h}_n^{(\delta_2-\delta_1, \delta_2-\delta_1)}(x - \delta_2; 1-2\delta_2),$$

when $-2\delta_2 \in \mathbf{N}$ or the symmetric Hahn polynomials [3, 28] as

$$S_n \left(\begin{array}{cc|c} -2 & 1+2(\delta_1+\delta_2) & x \\ -\delta_2-\delta_1(1+2\delta_2) & \delta_1\delta_2 & \end{array} \right) \\ = \frac{(\delta_1+\delta_2)_n (2\delta_1)_n}{(n+2(\delta_1+\delta_2)-1)_n} {}_3F_2 \left(\begin{array}{c|c} -n, n+2(\delta_1+\delta_2)-1, \delta_1-x & 1 \\ \delta_1+\delta_2, 2\delta_1 & \end{array} \right) \\ = h_n^{(\delta_1+\delta_2-1, \delta_1+\delta_1-1)}(x - \delta_1; 1-2\delta_1),$$

when $-2\delta_1 \in \mathbf{N}$. Finally, for $\delta_1 + \delta_2 = 1$ we get to Gram polynomials [13].

REFERENCES

- [1] R. Archibald, K. Chen, A. Gelb, and R. Renaut, *Improving tissue segmentation of human brain mri through preprocessing by the gegenbauer reconstruction method*, NeuroImage **20** (2003), no. 1, 489 – 502.
- [2] I. Area, E. Godoy, A. Ronveaux, and A. Zarzo, *Corrigendum: “Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: discrete case”*, J. Comput. Appl. Math. **89** (1998), no. 2, 309–325.

- [3] ———, *Classical symmetric orthogonal polynomials of a discrete variable*, Integral Transforms Spec. Funct. **15** (2004), no. 1, 1–12.
- [4] N. M. Atakishiyev, L. E. Vicent, and K. B. Wolf, *Continuous vs. discrete fractional Fourier transforms*, J. Comput. Appl. Math. **107** (1999), no. 1, 73–95.
- [5] N. M. Atakishiyev and K. B. Wolf, *Approximation on a finite set of points through Kravchuk functions*, Rev. Mexicana Fís. **40** (1994), no. 3, 366–377.
- [6] ———, *Fractional Fourier-Kravchuk transform*, J. Opt. Soc. Amer. A **14** (1997), no. 7, 1467–1477.
- [7] D. Chakraborty and J. H. Jung, *Efficient determination of the critical parameters and the statistical quantities for Klein-Gordon and sine-Gordon equations with a singular potential using generalized polynomial chaos methods*, J. Comput. Sci. (2012), in press.
- [8] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York, 1978, Mathematics and its Applications, Vol. 13.
- [9] I. Dreissigacker and M. Lein, *Quantitative theory for the lateral momentum distribution after strong-field ionization*, Chem. Phys. (2012), in press.
- [10] D. Gottlieb and C.-W. Shu, *On the Gibbs phenomenon and its resolution*, SIAM Rev. **39** (1997), no. 4, 644–668.
- [11] S. Gottlieb, J.-H. Jung, and S. Kim, *A review of David Gottlieb's work on the resolution of the Gibbs phenomenon*, Commun. Comput. Phys. **9** (2011), no. 3, 497–519.
- [12] T. Hakioglu and K. B. Wolf, *The canonical Kravchuk basis for discrete quantum mechanics*, J. Phys. A **33** (2000), no. 16, 3313–3323.
- [13] F. B. Hildebrand, *Introduction to numerical analysis*, second ed., Dover Publications Inc., New York, 1987.
- [14] K. M. Hosny, *Fast computation of accurate Gaussian-Hermite moments for image processing applications*, Digit. Signal Process. **22** (2012), no. 3, 476 – 485.
- [15] A. Jirari, *Second-order Sturm-Liouville difference equations and orthogonal polynomials*, Mem. Am. Math. Soc. **542** (1995), 138 p.
- [16] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q -analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010, With a foreword by Tom H. Koornwinder.
- [17] O. E. Lancaster, *Orthogonal polynomials defined by difference equations*, Amer. J. Math. **63** (1941), no. 1, pp. 185–207.
- [18] P. Maroni and M. Mejri, *The symmetric D_ω -semi-classical orthogonal polynomials of class one.*, Numer. Algorithms **49** (2008), no. 1-4, 251–282.
- [19] M. Masjed-Jamei, *A basic class of symmetric orthogonal polynomials using the extended Sturm-Liouville theorem for symmetric functions*, J. Math. Anal. Appl. **325** (2007), no. 2, 753–775.
- [20] ———, *A generalization of classical symmetric orthogonal functions using a symmetric generalization of Sturm-Liouville problems*, Integral Transforms Spec. Funct. **18** (2007), no. 11-12, 871–883.
- [21] ———, *A basic class of symmetric orthogonal functions using the extended Sturm-Liouville theorem for symmetric functions*, J. Comput. Appl. Math. **216** (2008), no. 1, 128–143.
- [22] ———, *A basic class of symmetric orthogonal functions with six free parameters*, J. Comput. Appl. Math. **234** (2010), no. 1, 283–296.
- [23] M. Masjed-Jamei and I. Area, *A symmetric generalization of Sturm-Liouville problems in discrete spaces*, submitted, 2012.
- [24] M. Masjed-Jamei and M. Dehghan, *A generalization of Fourier trigonometric series*, Comput. Math. Appl. **56** (2008), no. 11, 2941–2947.
- [25] M. Masjed-Jamei and W. Koepf, *On incomplete symmetric orthogonal polynomials of Jacobi type*, Integral Transforms Spec. Funct. **21** (2010), no. 9-10, 655–662.
- [26] ———, *On incomplete symmetric orthogonal polynomials of Laguerre type.*, Appl. Anal. **90** (2011), no. 3-4, 769–775.
- [27] M.-S. Min, T.-W. Lee, P. F. Fischer, and Stephen K. Gray, *Fourier spectral simulations and gegenbauer reconstructions for electromagnetic waves in the presence of a metal nanoparticle*, J. Comput. Phys. **213** (2006), no. 2, 730 – 747.
- [28] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991.

- [29] A. F. Nikiforov and V. B. Uvarov, *Special functions of mathematical physics. a unified introduction with applications*, Birkhäuser Verlag, Basel, 1988.
- [30] A. Ronveaux, S. Belmehdi, E. Godoy, and A. Zarzo, *Recurrence relation approach for connection coefficients. Applications to classical discrete orthogonal polynomials*, Symmetries and integrability of difference equations (Estérel, PQ, 1994), CRM Proc. Lecture Notes, vol. 9, Amer. Math. Soc., Providence, RI, 1996, pp. 319–335.
- [31] A. Ronveaux, A. Zarzo, and E. Godoy, *Recurrence relations for connection coefficients between two families of orthogonal polynomials*, J. Comput. Appl. Math. **62** (1995), no. 1, 67–73.
- [32] S. K. Suslov, *On the theory of difference analogues of special functions of hypergeometric type*, Uspekhi Mat. Nauk **44** (1989), no. 2(266), 185–226.
- [33] H. Zhu, *Image representation using separable two-dimensional continuous and discrete orthogonal moments*, Pattern Recognition **45** (2012), no. 4, 1540 – 1558.

(Masjed-Jamei) DEPARTMENT OF MATHEMATICS, K.N.TOOSI UNIVERSITY OF TECHNOLOGY, P.O. BOX 16315–1618, TEHRAN, IRAN.

E-mail address, Masjed-Jamei: `mmjamei@kntu.ac.ir`, `mmjamei@yahoo.com`

(Area) DEPARTAMENTO DE MATEMÁTICA APLICADA II, E.E. DE TELECOMUNICACIÓN, UNIVERSIDADE DE VIGO, CAMPUS LAGOAS-MARCOSENDE, 36310 VIGO, SPAIN.

E-mail address, Area: `area@uvigo.es`